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Asymptotic expansion of the $\mathcal{N} = 4$ dyon degeneracy

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Abstract: We study various aspects of power suppressed as well as exponentially suppressed corrections in the asymptotic expansion of the degeneracy of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories. In particular we explicitly calculate the power suppressed corrections up to second order and the first exponentially suppressed corrections. We also propose a macroscopic origin of the exponentially suppressed corrections using the quantum entropy function formalism. This suggests a universal pattern of exponentially suppressed corrections to all extremal black hole entropies in string theory.

Keywords: Black Holes in String Theory, AdS-CFT Correspondence

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Contents

1 Introduction and summary

One of the major successes of string theory has been the matching of the Bekenstein-Hawking entropy of a class of extremal black holes and the statistical entropy of a system of branes carrying the same quantum numbers as the black hole [1]. The initial comparison between the two was done in the limit of large charges. In this limit the analysis simplifies on both sides. On the gravity side we can restrict our analysis to two derivative terms in the action, while on the statistical side the analysis simplifies because we can use certain asymptotic formula to estimate the degeneracy of states for large charges. However given the successful matching between the statistical entropy and Bekenstein-Hawking entropy in the large charge limit, it is natural to explore whether the agreement continues to hold beyond this approximation. On the gravity side this requires taking into account the effect of higher derivative corrections and quantum corrections in computing the entropy. The effect of higher derivative terms is captured by the Wald's generalization of the Bekenstein-Hawking formula [2]. For extremal black holes this leads to the entropy function formalism for computing the entropy [3]. Recently it has been suggested that the effect of quantum corrections to the entropy of extremal black holes is encoded in the quantum entropy function, defined as the partition function of string theory on the near horizon geometry of the black holes [4]. On the other hand computing higher derivative corrections to the statistical entropy requires us to compute microscopic degeneracies of the black hole to greater accuracy. Here significant progress has been made in a class of $\mathcal{N} = 4$ supersymmetric field theories, for which we now have exact formulæ for the microscopic degeneracies [5–29]. (For a similar proposal in $\mathcal{N} = 2$ supersymmetric theories, see [30].)

Our eventual goal is to compare the statistical entropy computed from the exact degeneracy formula to the predicted result on the black hole side from the computation of the quantum entropy function (or whatever formula gives the exact result for the entropy

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of extremal black holes). However in practice we can compute the black hole side of the result only as an expansion in inverse powers of charges, by matching these to an expansion in powers of derivatives / string coupling constant. Thus we must carry out a similar expansion of the statistical entropy if we want to compare the results on the two sides. A systematic procedure for developing such an expansion of the statistical entropy has been discussed in [5, 6, 10, 13]. Our main goal in this paper is to explore this expansion in more detail, and. to whatever extent possible, relate it to the results of macroscopic computation.

The rest of the paper is organized as follows. In § 2 we give a brief overview of the exact dyon degeneracy formula in a class of $\mathcal{N} = 4$ supersymmetric string theories, and discuss the systematic procedure of extracting the degeneracy for large but finite charges. We also organise the computation of the statistical entropy by representing the result as a sum of contributions from single centered and multi-centered black holes, and then express the single centered black hole entropy as an asymptotic expansion in inverse powers of charges, together with exponentially suppressed corrections. In § 3 we examine the leading exponential term in the expression for the statistical entropy and compute the statistical entropy to order $1/\text{charge}^2$. Previous computation of the statistical entropy was carried out to order charge^0. We compare these results with the exact result for the statistical entropy and find good agreement. We also find that the agreement is worse if we compare the result with the exact statistical entropy in a domain where besides single centered black holes, we also have contribution from two centered black holes. This confirms that the asymptotic expansion is best suited for computing the entropy of single centered black holes. From the gravity perspective these corrections should be captured by six derivative corrections to the effective action; however explicit analysis of such contributions has not been carried out so far.

In § 4 we analyze the contribution from the exponentially subleading terms to the entropy of single centered black holes. While power suppressed corrections to the statistical entropy have been compared to the higher derivative corrections to the black hole entropy in various approximations, so far there has been no explanation of these exponentially suppressed terms from the black hole side.[1] In § 5 we suggest a macroscopic origin of the exponentially suppressed contributions to the entropy from quantum entropy function formalism. In this formalism the leading contribution to the macroscopic degeneracy comes from path integral over the near horizon AdS_2 geometry of the black hole with appropriate boundary condition. We show that for the same boundary conditions there are other saddle points which have different values of the euclidean action. These values have precisely the form needed to reproduce the exponentially suppressed contributions to the leading microscopic degeneracy.

2 An overview of statistical entropy function

In this section, we briefly review the systematic procedure for computing the asymptotic expansion of the statistical entropy of a dyon in a class of $\mathcal{N} = 4$ supersymmetric string theories. The approach mainly follows [5, 6, 10, 13, 22]. Our notation will be that of [23].

[1] Note that this expansion is quite different from the Rademacher expansion studied in [31, 32] since we scale all the charges uniformly.

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2.1 Dyon degeneracy

Let us consider an $\mathcal{N}=4$ supersymmetric string theory with a rank r gauge group. We shall work at a generic point in the moduli space where the unbroken gauge group is $\mathrm{U}(1)^r$. The low energy supergravity describing this theory has a continuous $\mathrm{SO}(6,r-6)\times\mathrm{SL}(2,\mathbb{R})$ symmetry which is broken to a discrete subgroup in the full string theory. We denote by Q and P the r dimensional electric and magnetic charges of the theory, by L the $\mathrm{SO}(6,r-6)$ invariant metric and by $(Q^2,P^2,Q\cdot P)$ the combinations (Q^TLQ,P^TLP,Q^TLP). Then for a fixed set of values of discrete T-duality invariants the degeneracy $d(Q,P)$, – or more precisely the sixth helicity trace B_6 [33] — of a dyon carrying charges (Q,P) is given by a formula of the form:

$$d(Q,P)=(-1)^{Q\cdot P+1}\,\frac{1}{a_1a_2a_3}\int_{\mathcal{C}} d\check{\rho}\,d\check{\sigma}\,d\check{v}\,e^{-\pi i(\check{\rho}P^2+\check{\sigma}Q^2+2\check{v}Q\cdot P)}\,\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})}\,, \tag{2.1}$$

where $\check{\rho}\equiv\check{\rho}_1+i\check{\rho}_2$, $\check{\sigma}\equiv\check{\sigma}_1+i\check{\sigma}_2$ and $\check{v}\equiv\check{v}_1+i\check{v}_2$ are three complex variables, $\check{\Phi}$ is a function of $(\check{\rho},\check{\sigma},\check{v})$ which we shall refer to as the inverse of the dyon partition function, and $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labeled by $(\check{\rho},\check{\sigma},\check{v})$, given by

$$\begin{aligned}&\check{\rho}_2=M_1, &&\check{\sigma}_2=M_2, &&\check{v}_2=M_3,\\ &0\le\check{\rho}_1\le a_1, &&0\le\check{\sigma}_1\le a_2, &&0\le\check{v}_1\le a_3\,.\end{aligned} \tag{2.2}$$

The periods a_1, a_2 and a_3 of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ are determined by the the quantization laws of Q^2, P^2 and $Q\cdot P$. M_1, M_2 and M_3 are large but fixed numbers. The choice of the M_i's depend on the domain of the asymptotic moduli space in which we want to compute $d(Q,P)$. As we move from one domain to another crossing the walls of marginal stability, $d(Q,P)$ changes. However this change is captured completely by a deformation of the contour labelled by (M_1,M_2,M_3) without any change in the partition function $\check{\Phi}$ [17, 18]. A simple rule that expresses (M_1,M_2,M_3) in terms of the asymptotic moduli is [21]:

$$\begin{aligned}M_1&=\Lambda\left(\frac{|\lambda|^2}{\lambda_2}+\frac{Q_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right),\\ M_2&=\Lambda\left(\frac{1}{\lambda_2}+\frac{P_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right),\\ M_3&=-\Lambda\left(\frac{\lambda_1}{\lambda_2}+\frac{Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right),\end{aligned} \tag{2.3}$$

where Λ is a large positive number,

$$Q_R^2=Q^T(M+L)Q,\quad P_R^2=P^T(M+L)P,\quad Q_R\cdot P_R=Q^T(M+L)P\,, \tag{2.4}$$

$\lambda\equiv\lambda_1+i\lambda_2$ denotes the asymptotic value of the axion-dilaton moduli which belong to the gravity multiplet and M is the asymptotic value of the $r\times r$ symmetric matrix valued moduli field of the matter multiplet satisfying $MLM^T=L$.

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A special point in the moduli space is the attractor point corresponding to the charges (Q, P). If we choose the asymptotic values of the moduli fields to be at this special point then all multi-centered black hole solutions are absent and the corresponding degeneracy formula captures the degeneracies of single centered black hole only [21]. This attractor point corresponds to the choice of (M, λ) for which

$$Q_R^2 = 2Q^2, \quad P_R^2 = 2P^2, \quad Q_R \cdot P_R = 2Q \cdot P, \quad \lambda_2 = \frac{\sqrt{Q^2P^2 - (Q \cdot P)^2}}{P^2}, \quad \lambda_1 = \frac{Q \cdot P}{P^2}. \tag{2.5}$$

Substituting this into (2.3) we get

$$\begin{aligned} M_1 &= 2\,\Lambda \frac{Q^2}{\sqrt{Q^2P^2 - (Q \cdot P)^2}}, \\ M_2 &= 2\,\Lambda \frac{P^2}{\sqrt{Q^2P^2 - (Q \cdot P)^2}}, \\ M_3 &= -2\,\Lambda \frac{Q \cdot P}{\sqrt{Q^2P^2 - (Q \cdot P)^2}}. \end{aligned} \tag{2.6}$$

We can invert the Fourier integrals (2.1) by writing

$$d(Q, P) = (-1)^{Q \cdot P + 1}\, g\left(\frac{1}{2}P^2, \frac{1}{2}Q^2, Q \cdot P\right), \tag{2.7}$$

where $g(m, n, p)$ are the coefficients of Fourier expansion of the function $1/\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$:

$$\frac{1}{\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})} = \sum_{m,n,p} g(m, n, p)\, e^{2\pi i (m\,\check{\rho} + n\,\check{\sigma} + p\,\check{v})}. \tag{2.8}$$

Different choices of (M_1, M_2, M_3) in (2.1) will correspond to different ways of expanding $1/\check{\Phi}$ and will lead to different $g(m, n, p)$. Conversely, for $d(Q, P)$ associated with a given domain of the asymptotic moduli space, if we define $g(m.n, p)$ via eq. (2.7), then the choice of (M_1, M_2, M_3) is determined by requiring that the series (2.8) is convergent for $(\check{\rho}_2, \check{\sigma}_2, \check{v}_2) = (M_1, M_2, M_3)$.

A special case on which we shall focus much of our attention is the $\mathcal{N} = 4$ supersymmetric string theory obtained by compactifying type IIB string theory on $K3 \times T^2$ or equivalently heterotic string theory compactified on T^6. In this case the function $\check{\Phi}$ is given by the well known Igusa cusp form of weight 10 [34, 35]:

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) = \Phi_{10}(\check{\rho}, \check{\sigma}, \check{v}) = e^{2\pi i(\check{\rho} + \check{\sigma} + \check{v})} \prod_{\substack{k', l, j \in \mathbb{Z} \\ k', l \geq 0; j < 0 \text{ for } k' = l = 0}} \left(1 - e^{2\pi i(\check{\sigma} k' + \check{\rho} l + \check{v} j)}\right)^{c(4lk' - j^2)}, \tag{2.9}$$

where $c(u)$ is defined via the equation [36]

$$8\left[\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2}\right] = \sum_{j, n \in \mathbb{Z}} c(4n - j^2)\, e^{2\pi i n \tau + 2\pi i j z}. \tag{2.10}$$

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2.2 Asymptotic expansion and statistical entropy function

In order to compare the statistical entropy $S_{\rm stat}(Q,P) \equiv \ln d(Q,P)$ with the black hole entropy we need to extract the behaviour of $S_{\rm stat}(Q,P)$ for large charges. We shall now briefly review the strategy and the results. For details the reader is referred to [22].

1. Beginning with the expression for $d(Q,P)$ given in (2.1), we first deform the contour to small values of $(\check\rho_2, \check\sigma_2, \check v_2)$ (say of the order of 1/charge). In this case the contribution to $S_{\rm stat}$ from the deformed contour can be shown to be subleading, and hence the major contribution comes from the residue at the poles picked up by the contour during the deformation.

2. For any given pole, one of the three integrals in (2.1) can be done using residue theorem. The integration over the other two variables are carried out using the method of steepest descent. It turns out that in all known examples, the dominant contribution to $S_{\rm stat}$ computed using this procedure comes from the pole of the integrand i.e. zero of $\check\Phi$ at

$$\check\rho\check\sigma - \check v^2 + \check v = 0\,. \tag{2.11}$$

Furthermore near this pole $\check\Phi$ behaves as

$$\check\Phi(\check\rho,\check\sigma,\check v) \propto (2v-\rho-\sigma)^k\, v^2\, g(\rho)\, g(\sigma)\,, \tag{2.12}$$

where

$$\rho = \frac{\check\rho\check\sigma - \check v^2}{\check\sigma}, \qquad \sigma = \frac{\check\rho\check\sigma - (\check v -1)^2}{\check\sigma}, \qquad v = \frac{\check\rho\check\sigma - \check v^2 + \check v}{\check\sigma}\,, \tag{2.13}$$

k is related to the rank r of the gauge group via the relation

$$r = 2k+8\,, \tag{2.14}$$

and $g(\tau)$ is a known function which depends on the details of the theory. Typically it transforms as a modular function of weight $(k+2)$ under a certain subgroup of the $\mathrm{SL}(2,\mathbb{Z})$ group. In the (ρ,σ,v) variables the pole at (2.11) is at $v=0$. The constant of proportionality in (2.12) depends on the specific $\mathcal{N}=4$ string theory we are considering, but can be calculated in any given theory.

3. Using the residue theorem the contribution to the integral (2.1) from the pole at (2.11) can be brought to the form

$$e^{S_{\rm stat}(Q,P)} \equiv d(Q,P) \simeq \int \frac{d^2\tau}{\tau_2^2}\, e^{-F(\vec\tau)}\,, \tag{2.15}$$

where τ_1 and τ_2 are two complex variables, related to ρ and σ via

$$\rho \equiv \tau_1 + i\tau_2\ , \quad \sigma \equiv -\tau_1 + i\tau_2\,, \tag{2.16}$$

and

$$F(\vec{\tau}) = -\left[\frac{\pi}{2\tau_2}|Q-\tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) + \ln\left\{K_0\left(2(k+3)+\frac{\pi}{\tau_2}|Q-\tau P|^2\right)\right\}\right],$$
$$K_0 = constant\,. \tag{2.17}$$

Even though τ_1 and τ_2 are complex, we have used the notation $\tau = \tau_1 + i\tau_2$, $\bar{\tau} = \tau_1 - i\tau_2$, $|\tau|^2 = \tau\bar{\tau}$, and $|Q-\tau P|^2 = (Q-\tau P)(Q-\bar{\tau}P)$. Note that $F(\vec{\tau})$ also depends on the charge vectors (Q,P), but we have not explicitly displayed these in its argument. The $\simeq$ in (2.15) denotes equality up to the (exponentially subleading) contributions from the other poles.

4. We can analyze the contribution to (2.13) using the saddle point method. To leading order the saddle point corresponds to the extremum of the first term in the right hand side of (2.17). This gives

$$\tau_1 = \frac{Q\cdot P}{P^2}, \qquad \tau_2 = \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}\,. \tag{2.18}$$

Using (2.13), (2.16) we get

$$(\check{\rho},\check{\sigma},-\check{v}) = \frac{i}{2\sqrt{Q^2P^2-(Q\cdot P)^2}}(Q^2,P^2,Q\cdot P) - \left(0,0,\frac{1}{2}\right). \tag{2.19}$$

We can regard the result for $-S_{\rm stat}$ as the extremal value of the 1PI effective action in the zero dimensional quantum field theory, with fields $\tau,\bar{\tau}$ (or equivalently τ_1, τ_2) and action $F(\vec{\tau}) - 2\ln\tau_2$. A manifestly duality invariant procedure for evaluating $S_{\rm stat}$ was given in [13] using background field method and Riemann normal coordinates. The final result of this analysis is that $S_{\rm stat}$ is given by

$$S_{\rm stat} \simeq -\Gamma_B(\vec{\tau}_B) \qquad \text{at} \qquad \frac{\partial\Gamma_B(\vec{\tau}_B)}{\partial\vec{\tau}_B} = 0\,, \tag{2.20}$$

where $\Gamma_B(\vec{\tau}_B)$ is the sum of 1PI vacuum diagrams calculated with the action

$$\sum_{n=0}^{\infty}\frac{1}{n!}(\tau_{B2})^n\xi_{i_1}\ldots\xi_{i_n}\,D_{i_1}\cdots D_{i_n}F(\vec{\tau})\Big|_{\vec{\tau}=\vec{\tau}_B} - \ln\mathcal{J}(\vec{\xi})\,, \tag{2.21}$$

where

$$\mathcal{J}(\vec{\xi}) = \left[\frac{1}{|\xi|}\sinh|\xi|\right], \qquad |\xi| \equiv \sqrt{\bar{\xi}\xi}\,. \tag{2.22}$$

Here $\vec{\tau}_B$ is a fixed background value, ξ, $\bar{\xi}$ are zero dimensional quantum fields and

$$D_\tau(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) = (\partial_\tau - im/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})),$$
$$D_{\bar{\tau}}(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) = (\partial_{\bar{\tau}} + in/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau}))\,, \tag{2.23}$$

for any arbitrary ordering of D_τ and $D_{\bar{\tau}}$ in $D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})$.

This finishes the required background for generating the asymptotic expansion of the statistical entropy to any given order in inverse powers of charges, — all we need is to compute $\Gamma_B(\tau_B)$ to the desired order and then find its value at the extremum. The function $-\Gamma_B(\tau_B)$ is called the statistical entropy function.

2.3 Exponentially suppressed corrections

In our analysis we shall also be interested in studying the exponentially subleading contribution to the statistical entropy. These come from picking up the residues at the other zeroes of $\check{\Phi}$. The details of the analysis has been reviewed in [22]; here we summarize the results for the special case of heterotic string theory on T^6 [5]. In this case $k = 10$, $\check{\Phi}$ is given by the Siegel modular form Φ_{10}, and the periods (a_1, a_2, a_3) are all equal to 1. Φ_{10} has second order zeroes at

$$\begin{aligned} & n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - m_1\check{\rho} + m_2 = 0, \\ \text{for} \quad & m_1, n_1, m_2, n_2 \in \mathbb{Z},\ j \in 2\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}. \end{aligned} \tag{2.24}$$

Since eqs. (2.24) are invariant under $(\vec{m}, \vec{n}, j) \to (-\vec{m}, -\vec{n}, -j)$, we can use this symmetry to set $n_2 \geq 0$. For any given $n_2 \geq 1$ we can use the symmetry of Φ_{10} under integer shifts in $(\check{\rho}, \check{\sigma}, \check{v})$ to bring m_1, n_1 and j in the range

$$0 \leq n_1 \leq n_2 - 1, \qquad 0 \leq m_1 \leq n_2 - 1, \qquad 0 \leq j \leq 2n_2 - 1\,. \tag{2.25}$$

Using this symmetry we can fix (m_1, n_1, j) in this range, but then we must extend the integration range over (ρ_1, σ_1, v_1) to be over the whole real axes. For given n_2, m_1, n_1, j, the last equation in (2.24) then determines m_2 in terms of the other variables. This equation also forces j to be odd, and $m_1 n_1 + (j^2 - 1)/4$ to be an integer multiple of n_2. We can now evaluate the contribution from each of these poles using saddle point method. To leading order the location of the saddle point from the pole associated with a given set of values of m_i, n_i and j is given by [5, 22]

$$(\check{\rho}, \check{\sigma}, -\check{v}) = \frac{i}{2n_2\sqrt{Q^2P^2 - (Q\cdot P)^2}}(Q^2, P^2, Q\cdot P) - \frac{1}{n_2}\left(n_1, -m_1, \frac{j}{2}\right). \tag{2.26}$$

For $n_2 = 1$ we can choose $n_1 = m_1 = 0$, $j = 1$ and (2.26) reduces to (2.11).

Besides these there are also contributions from the poles corresponding to $n_2 = 0$. These are in fact the poles responsible for the jump in the degeneracy as we cross walls of marginal stability [17]. In particular for the wall associated with a decay of the form

$$(Q, P) \to (Q_1, P_1) + (Q_2, P_2)\,, \tag{2.27}$$

$$(Q_1, P_1) = (\alpha Q + \beta P, \gamma Q + \delta P), \qquad (Q_2, P_2) = (\delta Q - \beta P, -\gamma Q + \alpha P)\,, \tag{2.28}$$

$$\alpha\delta = \beta\gamma, \qquad \alpha + \delta = 1\,, \tag{2.29}$$

the jump in the index is given by the residue at the pole at

$$\check{\rho}\gamma - \check{\sigma}\beta + \check{v}(\alpha - \delta) = 0\,. \tag{2.30}$$

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Unlike the residues from the poles at (2.24), which grow as exponentials of quadratic powers of charges, the residues at the poles at (2.30) grow as exponentials of linear powers of charges. Thus one expects them to be suppressed compared to the contribution from all other poles of the form given in (2.24). Nevertheless we shall see that for small charges the residues at (2.30) give substantial subleading contribution to the statistical entropy.

2.4 Organising the asymptotic expansion

Consider the contour integral given in (2.1) with (M_1, M_2, M_3) given as in (2.3). In order to find the asymptotic expansion of this expression we need to deform the contour so that it passes through the saddle point. Since the integral is done over the real parts of $(\check\rho, \check\sigma, \check v)$ keeping their imaginary parts fixed, we shall deform the contour by varying the imaginary parts $(\check\rho_2, \check\sigma_2, \check v_2)$ of $(\check\rho, \check\sigma, \check v)$. For this we first note that in the $(\check\rho_2, \check\sigma_2, \check v_2)$ space, the point (M_1, M_2, M_3) given in (2.6) corresponding to the choice of the contour for single centered black holes, and the values of $(\check\rho_2, \check\sigma_2, \check v_2)$ given in (2.26) corresponding to various saddle points, lie along a straight line passing through the origin:

$$\frac{\check\rho_2}{Q^2} = \frac{\check\sigma_2}{P^2} = -\frac{\check v_2}{Q \cdot P}\,. \tag{2.31}$$

Thus we can first deform the contour from its initial position to the position (2.6), keeping $Im(\check\rho, \check\sigma, \check v)$ large all through, and then deform it along a straight line towards the origin. In the first step we shall only cross the poles of the type given in (2.30). This picks up the contribution to the entropy from the multi-centered black holes which were present at the point in the moduli space where we are computing the entropy. In the second stage we pick up the contribution from all the saddle points with $n_2 \geq 1$, but do not cross any pole of the type given in (2.30). These can then be regarded as the contribution to the entropy of a pure single centered black hole. Thus we see that the complete contribution to single centered black hole entropy comes from residues at the poles (2.24) with $n_2 \geq 1$. This suggests that at least for finite values of charges where the jumps across the walls of marginal stability are not extremely small compared to the total index, the asymptotic expansion, based on the residues at the poles at (2.24) with $n_2 \geq 1$, is better suited for reproducing the entropy of single centered black holes than that of single and multi-centered black holes together. We shall see this explicitly in our numerical analysis.

3 Power suppressed corrections

In § 2 we outlined a general procedure for computing the statistical entropy as an expansion in inverse powers of charges. In this section we shall use this method to compute the statistical entropy to order $1/q^2$ where q stands for a generic charge. For comparison we note that the leading correction to the entropy is quadratic in the charges. Contribution to $S_{\rm stat}$ up to order q^0 has been computed in [6, 10, 13].

We begin with the expression for $F(\vec\tau)$ given in (2.17) and carry out the background field expansion as described in (2.21). For this we organise (2.21) as a sum of three terms

$$F(\vec\tau) - \ln \mathcal{J}(\vec\xi) = F_0 + F_1 + F_2 \tag{3.1}$$

where

$$
\begin{aligned}
F_0 &= -\frac{\pi}{2\tau_2}\,|Q-\tau P|^2\,, \\
F_1 &= \ln g(\tau) + \ln g(-\bar\tau) + (k+2)\ln(2\tau_2) - \ln \mathcal{J}(\vec\xi) - \ln\left[K_0\frac{\pi}{\tau_2}|Q-\tau P|^2\right], \\
F_2 &= -\ln\left[1+\frac{2(k+3)\tau_2}{\pi|Q-\tau P|^2}\right], \qquad (3.2)
\end{aligned}
$$

represent respectively the leading piece of order q^2, the $O(q^0)$ piece and all terms of the order $q^{-2n}, n \geq 1$. Since the loop expansion is an expansion in powers of q^{-2}, in order to carry out a systematic expansion in powers of q^{-2} we need to regard F_0 as the tree level contribution, F_1 as the 1-loop contribution and F_2 as two and higher loop contributions. To compute Γ_B up to a certain order, we need to compute 1PI vacuum diagrams in the zero dimensional field theory with action $(F_0 + F_1 + F_2)$ up to that order regarding ξ as fundamental field. Thus for example in order to compute the contribution to Γ_B to order q^{-2} we need to include all one and two loop diagrams involving vertices from F_0, all one loop diagrams involving a single vertex of F_1 and the tree level contribution from F_0, F_1 and F_2.

To see more explicitly how the powers of q appear, we expand $F(\vec\tau)$ in field variable ξ around the background point $\vec\tau_B$. We then identify the quadratic term in ξ in the leading action F_0 with the inverse propagator and all other terms (including quadratic terms in the expansion of F_1 and F_2) as vertices. Since F_0 is of order q^2, this gives a propagator of order q^{-2}. All vertices coming from F_0 are of order q^2, all vertices coming from F_1 are of order q^0 and the vertices coming from F_2 are of order q^{-2n} with $n \geq 1$. Let us now consider a 1PI vacuum diagram with V_n number of n-th order vertices coming from F_0. Since there are no external legs, we have $\sum_n nV_n/2$ propagators. Thus the contribution from this diagram goes as

$$
q^{\sum_n (2-n)V_n}\,. \qquad (3.3)
$$

Similar counting works for vertices coming from F_1 and F_2, but every vertex coming from F_1 will carry an extra power of q^{-2} and every vertex coming from F_2 will carry two or more extra powers of q^{-2}. Thus an order q^{-2} contribution to the effective action can come from

$$
(V_4 = 1, \quad V_n = 0 \quad \text{for} \quad n \neq 4) \quad \text{or} \quad (V_3 = 2, \quad V_n = 0 \quad \text{for} \quad n \neq 3)\,, \qquad (3.4)
$$

if all the vertices are from F_0, and

$$
V_2 = 1, \quad V_n = 0 \quad \text{for} \quad n \neq 2\,, \qquad (3.5)
$$

if this single two point vertex is from F_1.[2] The possible diagrams associated with (3.4) have been shown in figure 1 whereas the diagram associated with (3.5) have been shown in figure 2. Finally the order q^{-2} contribution from F_2 is obtained by just adding the $F_2(\tau_B)$ term to $\Gamma_B(\tau_B)$.

[2] Note that F_0 does not give a two point vertex.

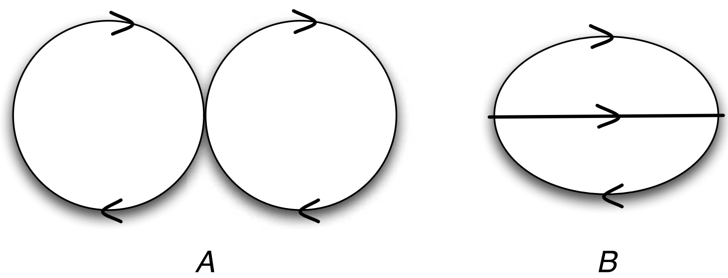

Figure 1. 2-loop graphs using the vertices from F_0.

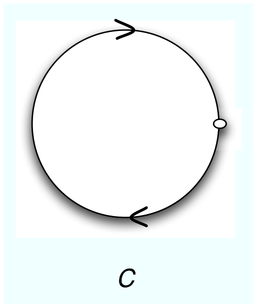

Figure 2. 1-loop graph using a 2-vertex from F_1.

The above analysis shows that in order to calculate the contribution to Γ_B up to order q^{-2}, we need to expand $F_0(\vec{\tau})$ to quartic order in $\vec{\xi}$, and $F_1(\vec{\tau})$ to quadratic order in $\vec{\xi}$. This is done with the help of (2.21), (2.23). We get[3]

$$\begin{aligned}
F_0(\vec{\tau}) &= F_0(\vec{\tau}_B) - \frac{i\pi}{4\tau_{B2}}\left\{\xi(Q-\bar{\tau}_B P)^2 - \bar{\xi}(Q-\tau_B P)^2\right\} - \frac{\pi}{4\tau_{B2}}\,|Q-\tau_B P|^2\,\bar{\xi}\xi \\
&\quad + \frac{i\pi}{24\tau_{B2}}\left\{(Q-\tau_B P)^2\bar{\xi}^2\xi - (Q-\bar{\tau}_B P)^2\xi^2\bar{\xi}\right\} - \frac{\pi}{48\tau_{B2}}|Q-\tau_B P|^2\,\bar{\xi}^2\xi^2\,, \\
F_1(\vec{\tau}) &= F_1(\vec{\tau}_B) + \tau_{B2}\left[\left\{\frac{g'(\tau_B)}{g(\tau_B)} + \frac{k+2}{\tau_B-\bar{\tau}_B} + \frac{1}{\tau_B-\bar{\tau}_B}\,\frac{(Q-\bar{\tau}_B P)^2}{|Q-\tau_B P|^2}\right\}\xi + c.c.\right] \\
&\quad - \left\{\frac{k+4}{4} - \frac{(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{4\left(|Q-\tau_B P|^2\right)^2} + \frac{1}{6}\right\}\xi\bar{\xi} + \mathcal{O}(\xi^2,\bar{\xi}^2)\,.
\end{aligned} \tag{3.6}$$

The quadratic term in the expansion of $F_0(\vec{\tau})$ gives the propagator

$$M^{\xi\bar{\xi}} = M^{\bar{\xi}\xi} = -\frac{4\tau_{B2}}{\pi|Q-\tau_B P|^2}, \qquad M^{\xi\xi} = M^{\bar{\xi}\bar{\xi}} = 0\,. \tag{3.7}$$

Using the vertices we can evaluate the order q^{-2} contribution to Γ_B shown in the three

[3]Whenever a τ (τ_B) appears without a vector sign, it should be interpreted as $\tau_1 + i\tau_2$ ($\tau_{B1} + i\tau_{B2}$).

diagrams in figures 1 and 2. The results are

$$
\begin{aligned}
A &= -\frac{2\tau_{B2}}{3\pi|Q-\tau_B P|^2}, \\
B &= \frac{2\tau_{B2}(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{9\pi(|Q-\tau_B P|^2)^3}, \\
C &= \frac{2\tau_{B2}}{3\pi|Q-\tau_B P|^2} + \frac{(4+k)\tau_{B2}}{\pi|Q-\tau_B P|^2} - \frac{\tau_{B2}(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{\pi\,(|Q-\tau_B P|^2)^3}\,.
\end{aligned} \tag{3.8}
$$

Combining this with the order q^2 and q^0 contribution to Γ_B given in [13], the complete statistical entropy function goes as,

$$
\begin{aligned}
\Gamma_B(\vec{\tau}_B) &= F_0(\vec{\tau}_B) + F_1(\vec{\tau}_B) + F_2(\vec{\tau}_B) - \ln(\pi\,|M^{\xi\bar{\xi}}|) + A + B + C \\
&= \Gamma_0(\vec{\tau}_B) + \Gamma_1(\vec{\tau}_B) + \Gamma_2(\vec{\tau}_B) \\
\Gamma_0(\vec{\tau}_B) &= -\frac{\pi}{2\tau_{B2}}\,|Q-\tau_B P|^2, \\
\Gamma_1(\vec{\tau}_B) &= \ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k+2)\ln(2\tau_{B2}) - \ln(4\pi K_0) \\
\Gamma_2(\vec{\tau}_B) &= -\frac{\tau_{B2}}{\pi|Q-\tau_B P|^2}\left((k+2) + \frac{7}{9}\frac{(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{(|Q-\tau_B P|^2)^2}\right).
\end{aligned} \tag{3.9}
$$

The last term in $\Gamma_2(\vec{\tau}_B)$ vanishes at the extremum of $\Gamma_0(\vec{\tau}_B)$ where

$$
\tau_{B2} = \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}, \quad \tau_{B1} = \frac{Q\cdot P}{P^2} \tag{3.10}
$$

We can therefore get rid of this term by doing a field redefinition. Using this we can write

$$
\Gamma_2(\vec{\tau}_B) = -\frac{\tau_{B2}}{\pi|Q-\tau_B P|^2}(k+2)\,. \tag{3.11}
$$

We now note that $\Gamma_2(\vec{\tau}_B)$ is independent of the modular form $g(\tau)$. This fact has some important implications for our result; we will come back to it at the end of this section.

We can now extremize $\Gamma_B(\vec{\tau}_B)$ given in (3.9) with respect to $\vec{\tau}_B$ to evaluate the black-hole entropy up to this order. For this it is enough to find the location of the extremum to order $1/q^2$. Let $\vec{\tau}_{(0)}$ be the extremum of $F_0(\vec{\tau}_B)$ given in (3.10). By extremizing F_0+F_1 we can find the extremum to order $1/q^2$. We get

$$
\tau = \tau_{(0)} + \frac{2\sqrt{Q^2P^2-(Q\cdot P)^2}}{\pi(P^2)^2}\,\overline{\frac{\partial\Gamma_1}{\partial\tau}} + \mathcal{O}(1/q^4)\,, \tag{3.12}
$$

where the derivative of Γ_1 is taken at fixed $\bar{\tau}$. Substituting this in the argument of the Γ_i's we get

$$
S_{\text{stat}} = -\Gamma_0 - \Gamma_1 - \Gamma_2 = S^{(0)} + S^{(1)} + S^{(2)}, \tag{3.13}
$$

where

$$
\begin{aligned}
S^{(0)} &= \pi\sqrt{Q^2P^2-(Q\cdot P)^2} \\
S^{(1)} &= -\ln g(\tau_{(0)}) - \ln g(-\bar{\tau}_{(0)}) - (k+2)\ln(2\tau_{(0)2}) + \ln(4\pi K_0) \\
S^{(2)} &= \frac{2+k}{2\pi\sqrt{Q^2P^2-(Q\cdot P)^2}} + \left[\left(\frac{g'(\tau_{(0)})}{g(\tau_{(0)})} + \frac{k+2}{\tau_{(0)}-\bar{\tau}_{(0)}}\right)\left(\frac{g'(-\bar{\tau}_{(0)})}{g(-\bar{\tau}_{(0)})} + \frac{k+2}{\tau_{(0)}-\bar{\tau}_{(0)}}\right)\right] \\
&\quad \times \frac{4\tau_{(0)2}^3}{\pi|Q-\tau_{(0)}P|^2}\,.
\end{aligned} \tag{3.14}
$$

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S^{(0)}_{\text{stat}}$	$S^{(1)}_{\text{stat}}$	$S^{(2)}_{\text{stat}}$	D_1	D_2
2	2	0	50064	10.82	6.28	10.62	11.576	.2	-0.756
4	4	0	32861184	17.31	12.57	16.90	17.382	.41	-0.072
6	6	0	16193130552	23.51	18.85	23.19	23.506	.32	.004
8	8	0	7999169992704	29.71	25.13	29.47	29.71	.24	.000
10	10	0	4074192429737760	35.943	31.42	35.754	35.945	.189	-0.002
6	6	1	11232685725	23.14	18.59	22.88	23.15	.26	-0.01
6	6	2	4173501828	22.15	17.77	21.94	22.198	.21	-0.05
6	6	3	920577636	20.64	16.32	20.41	20.766	.23	-0.13
6	6	-1	11890608225	23.19	18.59	22.88	23.15	.31	.04
6	6	-2	2857656822	21.77	17.77	21.94	22.198	-0.17	-0.43
6	6	-3	2894345136	21.78	16.32	20.41	20.766	1.37	1.01

Table 1. Comparison of the exact statistical entropy to the tree level, one loop and two loop results obtained via the asymptotic expansion. In the last two columns D_1 is the difference of the exact result and the one loop result and D_2 is the difference of the exact result and the two loop result. We clearly see that for $Q \cdot P > 0$ where only single centered black holes contribute to S_{stat}, inclusion of the two loop results reduces the error, at least for large charges.

For type IIB string theory compactified on $K3 \times T^2$, $k = 10$, $g(\tau) = \eta(\tau)^{24}$ and $4\pi K_0 = 1$. We have shown in table 1 the approximate statistical entropies $S^{(0)}_{\text{stat}} = S^{(0)}$ calculated with the 'tree level' statistical entropy function, $S^{(1)}_{\text{stat}} = S^{(0)} + S^{(1)}$ calculated with the 'tree level' plus 'one loop' statistical entropy function and $S^{(2)}_{\text{stat}} = S^{(0)} + S^{(1)} + S^{(2)}$ calculated with the 'tree level' plus 'one loop' plus 'two loop' statistical entropy function and compared the results with the exact statistical entropy S_{stat}. The exact results for $d(Q,P)$ are computed using a choice of contour for which only single centered black holes contribute to the index for $Q \cdot P > 0$ and both single and 2-centered black hole solutions contribute for $Q \cdot P < 0$. We clearly see that the asymptotic expansion has better agreement with the exact results when only single centered black holes are present, in accordance with our general argument.

Given the result for the statistical entropy to this order, one would like to see if this can be reproduced from the macroscopic calculation on the black hole side. So far black hole entropy calculation has been done for the leading supergravity action and a subset of the four derivative terms which include curvature squared contribution to the effective action [37–40]. The results of these two completely independent calculations match up to order q^0 and give us enough confidence on the expected equivalence of the statistical

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entropy and the black hole entropy. However there are many open issues. Even at the level of the four derivative terms, only a subset of the four derivative terms have been included in the analysis of the black hole entropy. Furthermore at this order the full 1PI effective action of string theory also contains non-local terms from integrating out the massless fermions and Wald's formula cannot even be applied in principle to take into account the effect of these terms. Recently a generalization of the Wald's formula for extremal black holes in the full quantum theory has been proposed [4] (see also [41, 42]). This will be discussed in more detail in § 5 in the context of exponentially suppressed terms. However as far as the power law corrections are concerned, at present we do not have a complete calculation of the quantum entropy function for quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric theory even at the level of order q^0 terms. This prevents us from making a concrete statement on the agreement between the two entropies.[4]

Given that even at order q^0 we do not have a complete test of the equality between the microscopic and the macroscopic calculations, we cannot hope to have such a test for the order q^{-2} terms calculated here. However we can say a few words about the possible contributions on the macroscopic side which is needed to reproduce the order q^{-2} corrections to the statistical entropy. To this end we note that the order q^{-2} correction to the statistical entropy function $\Gamma_B(\vec{\tau}_B)$ given in (3.11) is manifestly invariant under continuous duality transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{R}\,. \tag{3.15}$$

Now while comparing the statistical entropy function to the black hole entropy function, the parameters τ get identified with the near horizon axion-dilaton modulus λ in the heterotic description [6, 10, 13]. This suggests that if the required correction comes from a local correction to the 1PI action, then the corresponding term must be invariant under a continuous S-duality transformation. Furthermore since we are looking for a correction of order q^{-2}, we require the correction to the Lagrangian density to be a six derivative term. This puts a strong restriction on the type of contribution to the local Lagrangian density that can be responsible for such corrections. We have not been able to find a candidate Lagrangian density. The most straightforward method for constructing duality invariant terms using Riemann tensors constructed out of canonical Einstein metric does not work since all such terms vanish in the $AdS_2 \times S^2$ near horizon geometry and hence do not contribute to the entropy function to this order. This of course does not rule out the existence of duality invariant terms constructed out of other fields. The other possibility

[4] It was shown in [9] that the leading asymptotic expansion of the entropy to all orders in inverse powers of charges, associated with the pole at (2.11), is consistent with the OSV formula [43] after inclusion of certain additional measure factors. Refs. [44–46] independently derived the same measure factor in the semiclassical approximation by requiring that the entropy is invariant under duality transformations. Our goal is to derive a general formula for the entropy of an extremal black hole based on some principle (like AdS/CFT) from which the results of [9, 44–46] would follow. In particular if one can establish that the asymptotic expansion of the quantum entropy function reduces to the formula given in [9, 44–46], this will automatically prove that the quantum entropy function agrees with the statistical entropy to all orders in inverse powers of charges.

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is that these contributions cannot be encoded in a local Lagrangian density, but come from the non-local contributions to the quantum entropy function arising from the path integral over string fields in the near horizon geometry. To this end we note that since the OSV formula reproduces the complete asymptotic expansion to all orders in q^{-2}, if we can derive the OSV formula from the quantum entropy function we shall automatically reproduce these corrections to the statistical entropy.

4 Exponentially suppressed corrections

In this section we shall analyze the exponentially suppressed contributions from the zeroes of Φ_{10} given in (2.24):

$$n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - m_1\check{\rho} + m_2 = 0\,, \tag{4.1}$$

with

$$m_1, n_1, m_2, n_2 \in \mathbb{Z},\ j \in 2\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \tag{4.2}$$

For this we define

$$\check{\Omega} = \begin{pmatrix} \check{\rho} & \check{v} \\ \check{v} & \check{\sigma} \end{pmatrix}\,, \tag{4.3}$$

and look for a symplectic transformation of the form:

$$\begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \equiv \Omega = (A\check{\Omega} + B)(C\check{\Omega} + D)^{-1}\,, \tag{4.4}$$

such that

$$v = \frac{n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - m_1\check{\rho} + m_2}{\det(C\check{\Omega} + D)}\,. \tag{4.5}$$

Here $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a 4×4 symplectic matrix. In this case (4.1) gets mapped to $v = 0$. On the other hand the modular transformation law of Φ_{10} gives

$$\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v}) = \{\det(C\check{\Omega} + D)\}^{-k}\, \Phi_{10}(\rho, \sigma, v)\,, \qquad k = 10\,. \tag{4.6}$$

Thus the behaviour of $\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v})$ near the zero (4.1) is given by

$$\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v}) = -\{\det(C\check{\Omega} + D)\}^{-k}\, 4\pi^2\, v^2\, g(\rho)\, g(v) + \mathcal{O}(v^4)\,, \qquad g(\rho) = \eta(\rho)^{24}\,. \tag{4.7}$$

We can now substitute (4.7) into (2.1) (with $\check{\Phi}$ replaced by Φ_{10}) and evaluate the integral over $\check{v}$ using residue theorem. For this we need to regard (ρ, σ, v) appearing in (4.7) as functions of $(\check{\rho}, \check{\sigma}, \check{v})$ via eq. (4.4), (4.5). The result is, up to a sign,

$$(-1)^{Q\cdot P} \int d\check{\rho}\, d\check{\sigma}\, e^{-\pi i(\check{\rho}P^2 + \check{\sigma}Q^2 + 2\check{v}Q\cdot P)}\, \det(C\check{\Omega} + D)^{k+2}\, (2n_2\check{v} - j)^{-2}$$
$$\times g(\rho)^{-1}\, g(\sigma)^{-1}\, \left(Q\cdot P + \mathcal{O}(1)\right)\,, \tag{4.8}$$

where $\check{v}$ and (ρ,σ) are to be regarded as functions of $(\check{\rho},\check{\sigma})$ via eqs. (4.1) and (4.4). The last factor in (4.8) proportional to $Q\cdot P$ comes from taking the derivative of the integrand other than the pole term with respect to $\check{v}$. We can now evaluate the $(\check{\rho},\check{\sigma})$ integral using the saddle point method. To leading order the location of the saddle point is obtained by extremizing the term in the exponent of (4.8) subject to the constraint (4.1). The result is given in eq. (2.26):

$$(\check{\rho},\check{\sigma},-\check{v}) = \frac{i}{2n_2\sqrt{Q^2P^2-(Q\cdot P)^2}}(Q^2,P^2,Q\cdot P) - \frac{1}{n_2}\left(n_1,-m_1,\frac{j}{2}\right). \tag{4.9}$$

The result of the integration over $(\check{\rho},\check{\sigma})$ can be expressed as

$$\begin{aligned}(-1)^{Q\cdot P}\Big[&\exp\left(-\pi i(\check{\rho}P^2+\check{\sigma}Q^2+2\check{v}Q\cdot P)\right)\,\det(C\check{\Omega}+D)^{k+2}\,(2n_2\check{v}-j)^{-2}\,g(\rho)^{-1}\,g(\sigma)^{-1}\\ &\times(Q\cdot P+\mathcal{O}(1))\,\left((\det\Delta)^{-1/2}+\mathcal{O}(1)\right)\Big]_{\rm saddle}\,,\end{aligned} \tag{4.10}$$

where the subscript 'saddle' denotes that we need to set $(\check{\rho},\check{\sigma},\check{v})$ to their saddle point values given in (4.9), and Δ is the 2×2 matrix:

$$\Delta = i\,Q\cdot P\begin{pmatrix}\partial^2\check{v}/\partial\check{\rho}^2 & \partial^2\check{v}/\partial\check{\rho}\partial\check{\sigma}\\ \partial^2\check{v}/\partial\check{\rho}\partial\check{\sigma} & \partial^2\check{v}/\partial\check{\sigma}^2\end{pmatrix}. \tag{4.11}$$

In evaluating (4.11) we need to regard $\check{v}$ as a function of $(\check{\rho},\check{\sigma})$ via eq. (4.1). Explicit computation gives

$$\det\Delta = (Q\cdot P)^2\,n_2^2/(2n_2\check{v}-j)^4\,. \tag{4.12}$$

Substituting this and (4.9) into (4.10) gives

$$\begin{aligned}&\frac{1}{n_2}\exp\left(\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/n_2\right)\,\left[\det(C\check{\Omega}+D)^{k+2}\,g(\rho)^{-1}\,g(\sigma)^{-1}\,(1+\mathcal{O}(q^{-2}))\right]_{\rm saddle}\\ &\quad\times(-1)^{Q\cdot P}\,\exp\left[i\pi(n_1P^2-m_1Q^2+jQ\cdot P)/n_2\right]\,,\end{aligned} \tag{4.13}$$

where we have how fixed the overall sign by requiring that it agrees with the result of [22] for $(m_1,n_1,n_2,m_2,j)=(0,0,1,0,1)$.

In order to evaluate the factor $\det(C\check{\Omega}+D)^{k+2}\,g(\rho)^{-1}\,g(\sigma)^{-1}$ appearing in (4.13) explicitly, we need to find explicitly the matrix $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ satisfying (4.5). We shall do this explicitly for $n_2=2$. In this case there are six possible values of $(\vec{m},\vec{n},j)$ consistent with (2.25), (4.2). They are

$$\begin{aligned}(m_1,n_1,m_2,n_2,j) = &\ (0,0,0,2,1),(1,0,0,2,1),(0,1,0,2,1),\\ &\ (0,0,-1,2,3),(1,0,-1,2,3),(0,1,-1,2,3)\,.\end{aligned} \tag{4.14}$$

Q^2	2	4	6	6	6	6
P^2	2	4	6	6	6	6
$Q \cdot P$	0	0	0	1	2	3
$\Delta d(Q,P)$	34.617	480.638	18537.1	20104.8	27652.3	0

Table 2. First exponentially suppressed contribution to $d(Q,P)$ and $S_{\rm stat}(Q,P)$. Note that the correction vanishes accidentally for $Q \cdot P = Q^2/2 = P^2/2$ odd.

In each of these cases we can find appropriate matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying (4.5). These transformations take the form:

$$\Omega = \begin{pmatrix} \frac{\check\rho}{(1-2\check v)^2-4\check\rho\check\sigma} & \frac{-2\check v^2+\check v+2\check\rho\check\sigma}{(1-2\check v)^2-4\check\rho\check\sigma} \\ \frac{-2\check v^2+\check v+2\check\rho\check\sigma}{(1-2\check v)^2-4\check\rho\check\sigma} & \frac{\check\sigma}{(1-2\check v)^2-4\check\rho\check\sigma} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \frac{\check\rho}{4(\check v-1)\check v+2\check\rho-4\check\rho\check\sigma+1} & \frac{-2\check v^2+\check v+\check\rho(2\check\sigma-1)}{4(\check v-1)\check v+2\check\rho-4\check\rho\check\sigma+1} \\ \frac{-2\check v^2+\check v+\check\rho(2\check\sigma-1)}{4(\check v-1)\check v+2\check\rho-4\check\rho\check\sigma+1} & \frac{2(\check v-1)\check v+\check\rho-2\check\rho\check\sigma+\check\sigma}{4(\check v-1)\check v+2\check\rho-4\check\rho\check\sigma+1} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \frac{-2(\check v-1)\check v+\check\rho+2\check\rho\check\sigma+\check\sigma}{(1-2\check v)^2-2(2\check\rho+1)\check\sigma} & \frac{-2\check v^2+\check v+2\check\rho\check\sigma+\check\sigma}{(1-2\check v)^2-2(2\check\rho+1)\check\sigma} \\ \frac{-2\check v^2+\check v+2\check\rho\check\sigma+\check\sigma}{(1-2\check v)^2-2(2\check\rho+1)\check\sigma} & \frac{\check\sigma}{(1-2\check v)^2-2(2\check\rho+1)\check\sigma} \end{pmatrix}, \quad \begin{pmatrix} \frac{\check\rho}{(\check v-1)^2-\check\rho\check\sigma} & \frac{1-\check v}{(\check v-1)^2-\check\rho\check\sigma}-2 \\ \frac{1-\check v}{(\check v-1)^2-\check\rho\check\sigma}-2 & \frac{\check\sigma}{(\check v-1)^2-\check\rho\check\sigma} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -\frac{(1-2\check v)^2-4\check\rho\check\sigma}{-2\check v+\check\rho+\check\sigma+1} & -\frac{\check v(2\check v-3)+\check\rho-2\check\rho\check\sigma+1}{-2\check v+\check\rho+\check\sigma+1} \\ -\frac{\check v(2\check v-3)+\check\rho-2\check\rho\check\sigma+1}{-2\check v+\check\rho+\check\sigma+1} & -\frac{\check v^2-(\check\rho+1)\check\sigma}{-2\check v+\check\rho+\check\sigma+1} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -\frac{\check v(3\check v-4)-\check\rho-3\check\rho\check\sigma-\check\sigma+1}{-2(\check v-1)\check v+\check\rho+2\check\rho\check\sigma+\check\sigma} & \frac{\check v-\check\rho-1}{-2(\check v-1)\check v+\check\rho+2\check\rho\check\sigma+\check\sigma}+1 \\ \frac{\check v-\check\rho-1}{-2(\check v-1)\check v+\check\rho+2\check\rho\check\sigma+\check\sigma}+1 & \frac{-2\check\rho-1}{-2(\check v-1)\check v+\check\rho+2\check\rho\check\sigma+\check\sigma}+2 \end{pmatrix}. \tag{4.15}$$

These transformations can be used to get ρ and σ in terms of $(Q^2, P^2, Q \cdot P)$ using (4.9). Substituting these into (4.13) and summing over the allowed values of (m_1, n_1, j) given in (4.14) we get the correction to $d(Q,P) = \exp(S_{\rm stat})$ to this order. If we denote the resulting correction to $d(Q,P)$ by $\Delta d(Q,P)$, then the values of $\Delta d(Q,P)$ for different values of $(Q^2, P^2, Q \cdot P)$ have been shown in table 2.

5 Macroscopic origin of the exponentially suppressed corrections

We have seen that the corrections to the leading contribution to the statistical entropy are of two types, power suppressed corrections which arise from expansion about the saddle point associated with pole (2.11), and exponentially suppressed corrections associated with the contribution from the residues at the other poles (2.24). Given that we have not been able to reproduce even the power suppressed corrections from the macroscopic side, it may seem futile to attempt to understand the exponentially suppressed corrections. However we shall now argue that quantum entropy function may provide a natural mechanism for understanding the exponentially suppressed corrections.

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We shall begin with a lightening review of the quantum entropy function. Let us consider an extremal black hole with an AdS_2 factor in the near horizon geometry. We shall regard string theory in this background as a two dimensional theory, treating all other directions as compact. The background fields describing the AdS_2 near horizon geometry has the form [47]

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right), \quad F^{(i)}_{\rm rt} = e_i, \quad \cdots \tag{5.1}$$

where $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$ are the gauge field strengths associated with two dimensional gauge fields $A^{(i)}_\mu$, v and e_i are constants and $\cdots$ denotes near horizon values of other fields. Under euclidean continuation

$$t = -i\theta\,, \tag{5.2}$$

we have

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad F^{(i)}_{r\theta} = -i\, e_i, \quad \cdots \tag{5.3}$$

Under a further coordinate change

$$r = \cosh\eta\ , \tag{5.4}$$

(5.3) takes the form

$$ds^2 = v\ \left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \qquad F^{(i)}_{\theta\eta} = ie_i\,\sinh\eta, \qquad \cdots\,.$$

The metric is non-singular at the point $\eta = 0$ if we choose θ to have period 2π. Integrating the field strength we can get the form of the gauge field:

$$A^{(i)}_\mu dx^\mu = -i\, e_i\,(\cosh\eta - 1)d\theta = -i\, e_i\,(r-1)d\theta\,. \tag{5.5}$$

Note that the -1 factor inside the parenthesis is required to make the gauge fields non-singular at $\eta = 0$. In writing (5.5) we have chosen $A^{(i)}_\eta = 0$ gauge. If q_i denotes the charge of the black hole corresponding to the ith gauge field and $\mathcal{L}$ denotes the Lagrangian density evaluated in the near horizon geometry (5.5), then $\vec{q}$ and $\vec{e}$ are related as

$$q_i = \frac{\partial(v\mathcal{L})}{\partial e_i}\,. \tag{5.6}$$

Quantum entropy function is a proposal for computing the exact degeneracy of states of an extremal black hole. It is given by

$$d(\vec{q}) = \left\langle \exp\left[-iq_i\oint d\theta\, A^{(i)}_\theta\right]\right\rangle^{\rm finite}_{AdS_2}, \tag{5.7}$$

where $\langle\ \rangle_{AdS_2}$ denotes the unnormalized path integral over various fields of string theory on euclidean global AdS_2 described in (5.5) and $A^{(i)}_\theta$ denotes the component of the i-th gauge field along the boundary of AdS_2. The superscript '$finite$' refers to the finite part of the amplitude defined as follows. If we regularize the infra-red divergence by putting an explicit cut-off that regularizes the volume of AdS_2, then the amplitude has the form $e^{CL}\times$

a finite part where C is a constant and L is the length of the boundary of regulated AdS_2. We define the finite part as the one obtained by dropping the e^{CL} part. This equation gives a precise relation between the microscopic degeneracy and an appropriate partition function in the near horizon geometry of the black hole.

In defining the path integral over AdS_2 we need to put boundary conditions on various fields. We require that the asymptotic geometry coincides with (5.5). Special care is needed to fix the boundary condition on $A^{(i)}_\theta$. In the $A^{(i)}_\eta = 0$ gauge the Maxwell's equation around this background has two independent solutions near the boundary: $A^{(i)}_\theta$ = constant and $A^{(i)}_\theta \propto r$. Since the latter is the dominant mode we put boundary condition on the latter mode, allowing the constant mode of the gauge field to fluctuate. This corresponds to working with fixed asymptotic values of the electric fields, or equivalently fixed charges via eq. (5.6).

Let us now review how in the classical limit the quantum entropy function reduces to the exponential of the Wald entropy. For this we need to put an infra-red cut-off; this is done by restricting the coordinate r in the range $1 \le r \le r_0$. Then in the classical limit the quantum entropy function is given by the finite part of

$$\exp\left(-A_{\mathrm{bulk}} - A_{\mathrm{boundary}} - iq_i \oint A^{(i)}_\theta \, d\theta\right), \tag{5.8}$$

where A_{bulk} and A_{boundary} represent contributions from the bulk and the boundary terms in the classical action in the background (5.5). If $\mathcal{L}$ denotes the Lagrangian density of the two dimensional theory, then the bulk contribution to the action in the background (5.5) takes the form:

$$\begin{aligned} A_{\mathrm{bulk}} &= -\int d^2x \sqrt{\det g}\, \mathcal{L} \\ &= -\int_0^{2\pi} d\theta \int_0^{\cosh^{-1} r_0} d\eta \, \sinh\eta \, v \, \mathcal{L} \\ &= -2\pi \, v \, \mathcal{L}\,(r_0 - 1) + \mathcal{O}(r_0^{-1}). \end{aligned} \tag{5.9}$$

In going from the second to the third step in (5.9) we have used the fact that due to the SO(2,1) invariance of the AdS_2 background, $\mathcal{L}$ must be independent of η and θ. In this parametrization the length L of the boundary is given by

$$L = \sqrt{v} \int_0^{2\pi} \sqrt{r_0^2 - 1}\, d\theta = 2\pi \sqrt{v}\, r_0 + \mathcal{O}(r_0^{-1}). \tag{5.10}$$

The contribution from the last term in (5.8) can also be calculated easily using the expression for $A^{(i)}_\theta$ given in (5.5). We get

$$iq_i \oint A^{(i)}_\theta \, d\theta = 2\pi \, \vec{q} \cdot \vec{e}(r_0 - 1). \tag{5.11}$$

Finally, the contribution from A_{boundary} can be shown to have the form [4]

$$A_{\mathrm{boundary}} = 2\pi r_0 \, K + \mathcal{O}(r_0^{-1}), \tag{5.12}$$

for some constant K. This gives

$$\exp\left(-A_{\rm bulk}-A_{\rm boundary}-iq_i\oint A_\theta^{(i)}\,d\theta\right)=\exp\left[-2\pi r_0(\vec{q}\cdot\vec{e}-v\,\mathcal{L}+K)+\mathcal{O}(r_0^{-1})\right]$$
$$\times\exp\left[2\pi(\vec{q}\cdot\vec{e}-v\,\mathcal{L})\right]\,. \tag{5.13}$$

Thus the quantum entropy function, given by the finite part of (5.13), takes the form

$$d(q)\simeq\exp\left[2\pi(\vec{q}\cdot\vec{e}-v\,\mathcal{L})\right]\,. \tag{5.14}$$

The right hand side of (5.14) is the exponential of the Wald entropy [3].[5] For the particular case of quarter BPS black holes in $\mathcal{N}=4$ supersymmetric string theories the leading contribution to (5.14) has the form

$$d(q)\simeq\exp\left(\pi\sqrt{Q^2P^2-(Q\cdot P)^2}\right)\,. \tag{5.15}$$

Quantum corrections to (5.14) can be of two types. First of all we can have fluctuations of the string field around the AdS_2 background (5.3). We expect this to produce power law corrections, but not change the exponent in (5.15) which is related to the finite part of the action in the AdS_2 background. The other class of corrections could come from picking altogether different classical solutions with the same asymptotic field configuration as the one given in (5.3). These could have different actions and hence give contributions with different exponential factors. Thus such corrections are the ideal candidates for producing exponentially subleading corrections to the degeneracy.

Can we identify classical solutions which could produce the subleading corrections discussed in § 4? To this end consider a $\mathbb{Z}_N$ quotient of the background (5.3) by the transformation

$$\theta\to\theta+\frac{2\pi}{N}\,. \tag{5.16}$$

If we denote by $(\widetilde{r},\widetilde{\theta})$ the coordinates of this new space then the solution may be expressed as

$$ds^2=v\left((\widetilde{r}^2-1)d\widetilde{\theta}^2+\frac{d\widetilde{r}^2}{\widetilde{r}^2-1}\right),\quad F^{(i)}_{\widetilde{r}\widetilde{\theta}}=-i\,e_i,\quad\cdots,\qquad\widetilde{\theta}\equiv\widetilde{\theta}+\frac{2\pi}{N}\,. \tag{5.17}$$

Since $\widetilde{\theta}$ has a different period than θ, this does not manifestly have the same asymptotic form as the solution (5.3). Let us now make a change of coordinates

$$r=\widetilde{r}/N,\quad\theta=N\widetilde{\theta}\,. \tag{5.18}$$

In this coordinate system the new metric takes the form:

$$ds^2=v\left((r^2-N^{-2})d\theta^2+\frac{dr^2}{r^2-N^{-2}}\right),\quad F^{(i)}_{r\theta}=-i\,e_i,\quad\cdots,\qquad\theta\equiv\theta+2\pi\,. \tag{5.19}$$

This has the same asymptotic behaviour as the original solution and hence is a potential saddle point that could contribute to the quantum entropy function. The action associated

[5] For the special case of two derivative actions this has also been noted recently in [48].

with this solution, with the cut-off $r \le r_0$, can be easily calculated. After removing the r_0 dependent piece we get the following classical contribution to the quantum entropy function[6]

$$\exp\left[2\pi(\vec{q}\cdot\vec{e} - v\,\mathcal{L})/N\right] = \exp\left(\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/N\right)\,. \tag{5.20}$$

This has precisely the right form as the exponentially subleading contributions described in § 4 if we identify N with the integer n_2 appearing there.

This however cannot be the complete story. From the form of the solution given in (5.17) it is clear that the the solution has a $\mathbb{Z}_N$ orbifold singularity of the type $\mathbb{R}^2/\mathbb{Z}_N$ at the origin $\widetilde{r}=1$. This is *a priori* a singular configuration and it is not clear if this is an allowed configuration in string theory. We resolve this difficulty by accompanying the $\mathbb{Z}_N$ action by an internal $\mathbb{Z}_N$ transformation

$$\phi \to \phi - \frac{2\pi}{N}\,, \tag{5.21}$$

where ϕ is the azimuthal coordinate of the sphere S^2 that is also part of the near horizon geometry of the black hole. If ψ denotes the polar angle on S^2 then the orbifold group has fixed points at $(\widetilde{r}=1,\psi=0)$ and $(\widetilde{r}=1,\psi=\pi)$. Thus the manifold is still singular but now the singularities are of the type $\mathbb{C}^2/\mathbb{Z}_N$, and these can certainly be resolved in string theory. Thus we conclude that the resulting configuration is non-singular. The classical action is not affected by the additional shifts in the ϕ coordinate and hence the contribution to the quantum entropy function continues to be given by (5.20).

There is however a new issue that we need to address. Now the identification $\theta\equiv\theta+2\pi$ changes to

$$(\theta,\phi)\equiv\left(\theta+2\pi,\phi-\frac{2\pi}{N}\right)\,. \tag{5.22}$$

Thus one needs to check if this is consistent with the asymptotic boundary conditions imposed on various fields. To this end we note that if we denote by $\mathcal{A}_\mu$ the two dimensional gauge field arising from the ϕ translation isometry, then the twisted boundary condition (5.23) is equivalent to switching on a Wilson line of the form

$$\oint \mathcal{A}_\theta\, d\theta = \frac{2\pi}{N}\,. \tag{5.23}$$

Now as discussed earlier, for all gauge fields the boundary conditions fix the electric field, or equivalently the charge, but the zero modes of the gauge fields are allowed to fluctuate. Here the charge associated with the gauge field $\mathcal{A}_\mu$ is the angular momentum [49] which has been taken to be zero. But there is no constraint on the Wilson line $\oint\mathcal{A}_\theta\,d\theta$. Thus we are instructed to integrate over different possible values of this Wilson line, and in that process pick up contribution from the different saddle points given in (5.19). This shows that there is no conflict between the asymptotic boundary conditions and the twist described in (5.22).

[6]This is easiest to derive in the $(\widetilde{r},\widetilde{\theta})$ coordinate system where the total action is $1/N$ times the action for the original AdS_2 background with r_0 replaced by $\widetilde{r}_0$. Since $\widetilde{r}_0 = Nr_0$, the terms linear in r_0 are the same as in the original AdS_2 background, whereas the r_0 independent term gets divided by N.

Another issue that needs attention is integration over bosonic and fermonic zero modes associated with this solution. The near horizon geometry of the black hole has an $\mathcal{N}=4$ superconformal algebra. The generators of this algebra are the SL$(2,R)$ generators L_0, $L_{\pm 1}$, the SU(2) generators J^3, $J^\pm$ and the supersymmetry generators $G^{\pm\alpha}_{\pm\frac{1}{2}}$. with $\alpha = 1,2$. Of these $(L_1 - L_{-1})/2$ is the generator of rotation about the origin of AdS_2 and J^3 is the generator of rotation about the north pole of S^2. Since the orbifold action is generated by $(L_1 - L_{-1} - 2J^3)$, the quotient is not invariant under the full $\mathcal{N}=4$ superconformal algebra; it is invariant only under a subalgebra that commutes with $(L_1 - L_{-1} - 2J^3)$. This subalgebra is generated by $L_1 - L_{-1}$, J^3, $G^{+\alpha}_{1/2} + G^{+\alpha}_{-1/2}$ and $G^{-\alpha}_{1/2} - G^{-\alpha}_{-1/2}$. The broken bosonic and fermionic generators leads to four bosonic and four fermionic zero modes of the solution. Of these the bosonic zero modes parametrize the coset $(\mathrm{SL}(2,R)/\mathrm{U}(1)) \times (\mathrm{SU}(2)/\mathrm{U}(1)) = AdS_2 \times S^2$. This is precisely the situation analyzed in [50].[7] Naively the integration over the bosonic zero modes will produce infinite result and the fermionic zero mode integrals vanish. But it was shown in [50] that we can regularize the inregrals by adding to the action an extra term that does not affect the integral. The extra term lifts both the bosonic and the fermionic zero modes and as a result the path integral produces a finite result.

There are several other minor issues which need to be addressed. For type II string theory in flat space-time, the $\mathbb{Z}_N$ orbifold action described here generates an allowed configuration. Here we have an $AdS_2 \times S^2$ background instead of flat space. Hence the original analysis is not strictly valid. However since the orbifold fixed point is localized in $AdS_2 \times S^2$, it should not 'feel' the effect of the background geomery and continue to be an allowed configuration. What is not guaranteed is that the blow up modes which allow us to deform the configuration away from the orbifold point will remain flat directions. This is an important issue we need to address if we want to explore the constant multiplying (5.20). We also need to explore if there can be any additional contribution to the action from the orbifold fixed point. We expect however that since the fixed point is localized at a point in $AdS_2 \times S^2$, to leading order such a contribution (if non-zero) will be independent of the background geometry of $AdS_2 \times S^2$. In particular it will not have a factor proportional to the size of $AdS_2 \times S^2$, and hence will at most give an order q^0 correction to the leading term $\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/N$ in the exponent of (5.20).

The analysis described above is independent of which kind of extremal black hole we are considering.[8] This suggests a universal pattern of the exponentially suppressed corrections to the entropy of all extremal black holes. If we denote by S_0 the leading contribution to the entropy then the exact degeneracy should contain subleading corrections of order $e^{S_0/N}$ for all $N \in \mathbb{Z}$, $N \geq 2$. It will be interesting to see if the exact degeneracy formulæ of extremal black holes in theories with less number of supersymmetries obey this structure.

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[7] The notation of [50] is slightly different; what we are calling $L_1 - L_{-1}$ was called L_0 in [50].

[8] For higher dimensional black holes the near horizon geometry contains a (squashed) S^n factor instead of S^2. In this case we can choose a suitable embedding of the $\mathbb{Z}_N$ action inside the symmetry group of (squashed) S^n.

References

[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029] [SPIRES].

[2] R.M. Wald, *Black hole entropy is the Nöther charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038] [SPIRES].

[3] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177] [SPIRES].

[4] A. Sen, *Quantum entropy function from AdS_2/CFT_1 correspondence*, arXiv:0809.3304 [SPIRES].

[5] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026] [SPIRES].

[6] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287] [SPIRES].

[7] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094] [SPIRES].

[8] D. Gaiotto, *Re-recounting dyons in $N = 4$ string theory*, hep-th/0506249 [SPIRES].

[9] D. Shih and X. Yin, *Exact black hole degeneracies and the topological string*, *JHEP* **04** (2006) 034 [hep-th/0508174] [SPIRES].

[10] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147] [SPIRES].

[11] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, *JHEP* **06** (2006) 064 [hep-th/0602254] [SPIRES].

[12] A. Dabholkar and S. Nampuri, *Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift*, *JHEP* **11** (2007) 077 [hep-th/0603066] [SPIRES].

[13] J.R. David and A. Sen, *CHL dyons and statistical entropy function from $D1 - D5$ system*, *JHEP* **11** (2006) 072 [hep-th/0605210] [SPIRES].

[14] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in $N = 4$ supersymmetric type-II string theories*, *JHEP* **11** (2006) 073 [hep-th/0607155] [SPIRES].

[15] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in generic $N = 4$ supersymmetric $Z(N)$ orbifolds*, *JHEP* **01** (2007) 016 [hep-th/0609109] [SPIRES].

[16] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, *JHEP* **12** (2007) 087 [hep-th/0612011] [SPIRES].

[17] A. Sen, *Walls of marginal stability and dyon spectrum in $N = 4$ supersymmetric string theories*, *JHEP* **05** (2007) 039 [hep-th/0702141] [SPIRES].

[18] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023 [hep-th/0702150] [SPIRES].

[19] N. Banerjee, D.P. Jatkar and A. Sen, *Adding charges to $N = 4$ dyons*, *JHEP* **07** (2007) 024 [arXiv:0705.1433] [SPIRES].

[20] A. Sen, *Two centered black holes and $N = 4$ dyon spectrum*, *JHEP* **09** (2007) 045 [arXiv:0705.3874] [SPIRES].

[21] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, *JHEP* **09** (2007) 070 [arXiv:0706.2363] [SPIRES].

[22] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [arXiv:0708.1270] [SPIRES].

[23] S. Banerjee, A. Sen and Y.K. Srivastava, *Generalities of quarter BPS dyon partition function and dyons of torsion two*, *JHEP* **05** (2008) 101 [arXiv:0802.0544] [SPIRES].

[24] S. Banerjee, A. Sen and Y.K. Srivastava, *Partition functions of torsion > 1 dyons in heterotic string theory on T^6*, *JHEP* **05** (2008) 098 [arXiv:0802.1556] [SPIRES].

[25] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, arXiv:0803.2692 [SPIRES].

[26] M.C.N. Cheng and E.P. Verlinde, *Wall crossing, discrete attractor flow and Borcherds algebra*, arXiv:0806.2337 [SPIRES].

[27] S. Govindarajan and K. Gopala Krishna, *Generalized Kac-Moody algebras from CHL dyons*, *JHEP* **04** (2009) 032 [arXiv:0807.4451] [SPIRES].

[28] S. Banerjee, A. Sen and Y.K. Srivastava, *Genus two surface and quarter BPS dyons: the contour prescription*, *JHEP* **03** (2009) 151 [arXiv:0808.1746] [SPIRES].

[29] M.C.N. Cheng and A. Dabholkar, *Borcherds-Kac-Moody symmetry of $N = 4$ dyons*, arXiv:0809.4258 [SPIRES].

[30] J.R. David, *On the dyon partition function in $N = 2$ theories*, *JHEP* **02** (2008) 025 [arXiv:0711.1971] [SPIRES].

[31] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole Farey tail*, hep-th/0005003 [SPIRES].

[32] J. Manschot and G.W. Moore, *A modern Farey tail*, arXiv:0712.0573 [SPIRES].

[33] A. Gregori et al., *R^2 corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys.* **B 510** (1998) 423 [hep-th/9708062] [SPIRES].

[34] J. Igusa, *On Siegel modular varieties of genus two*, *Amer. J. Math.* **84** (1962) 175.

[35] J. Igusa, *On Siegel modular varieties of genus two. II*, *Amer. J. Math.* **86** (1962) 392.

[36] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with* SU(N) *holonomy*, *Nucl. Phys.* **B 315** (1989) 193 [SPIRES].

[37] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [hep-th/9812082] [SPIRES].

[38] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [hep-th/9906094] [SPIRES].

[39] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortsch. Phys.* **49** (2001) 3 [hep-th/0007195] [SPIRES].

[40] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008 [hep-th/0508042] [SPIRES].

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[41] A. Sen, *Entropy function and AdS_2/CFT_1 correspondence*, JHEP **11** (2008) 075 [arXiv:0805.0095] [SPIRES].

[42] R.K. Gupta and A. Sen, *AdS_3/CFT_2 to AdS_2/CFT_1*, JHEP **04** (2009) 034 [arXiv:0806.0053] [SPIRES].

[43] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, Phys. Rev. **D 70** (2004) 106007 [hep-th/0405146] [SPIRES].

[44] B. de Wit, *Supersymmetric black hole partition functions*, talk given at *Strings* 2005, July 11-16, Toronto, Canada (2005).

[45] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Black hole partition functions and duality*, JHEP **03** (2006) 074 [hep-th/0601108] [SPIRES].

[46] G.L. Cardoso, B. de Wit and S. Mahapatra, *Subleading and non-holomorphic corrections to $N = 2$ BPS black hole entropy*, JHEP **02** (2009) 006 [arXiv:0808.2627] [SPIRES].

[47] J.M. Maldacena, J. Michelson and A. Strominger, *Anti-de Sitter fragmentation*, JHEP **02** (1999) 011 [hep-th/9812073] [SPIRES].

[48] A. Castro, D. Grumiller, F. Larsen and R. McNees, *Holographic description of AdS_2 black holes*, JHEP **11** (2008) 052 [arXiv:0809.4264] [SPIRES].

[49] D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, *Rotating attractors*, JHEP **10** (2006) 058 [hep-th/0606244] [SPIRES].

[50] C. Beasley et al., *Why $Z_{\mathrm{BH}} = |Z_{\mathrm{top}}|^2$*, hep-th/0608021 [SPIRES].